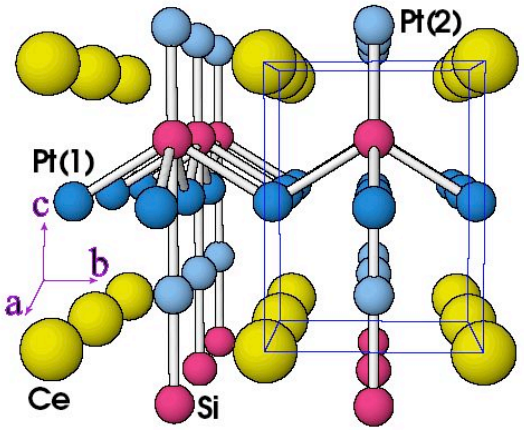
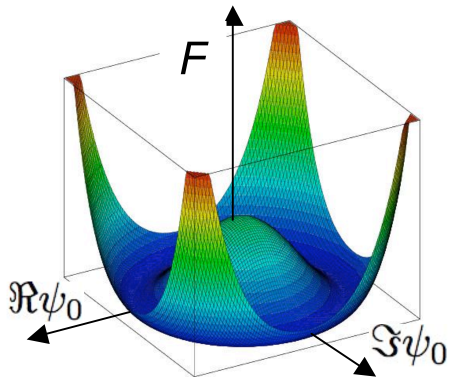
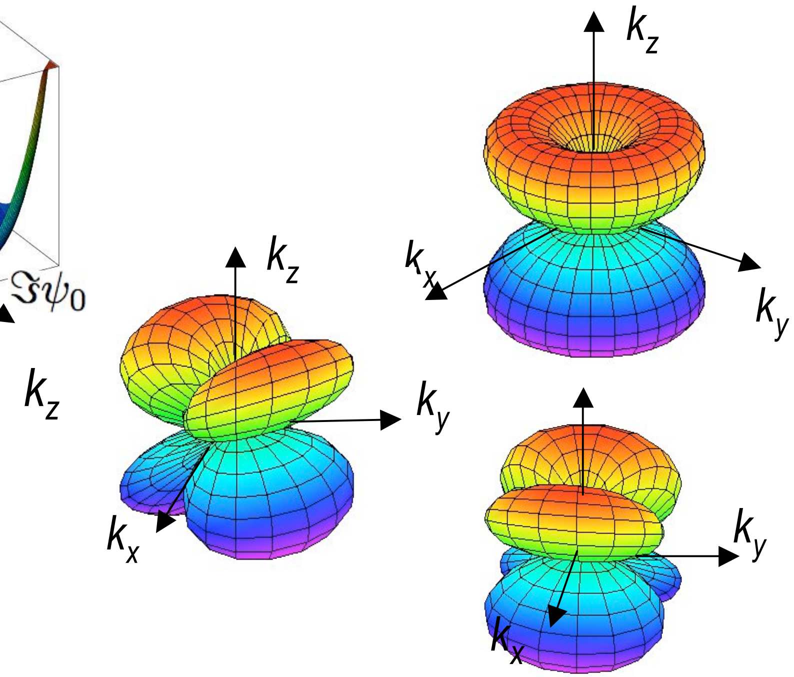
Symmetry aspects of Superconductivity

Manfred Sigrist *ETH Zurich*







Outline

Part 1

- superconductivity basics: thermodynamics, London theory
- Ginzburg-Landau theory: symmetry breaking order parameter
- properties of superconducting condensate
- BCS theory: basics to the quasiparticle gap
- Generalized BCS: towards unconventional superconductivity

Part 2

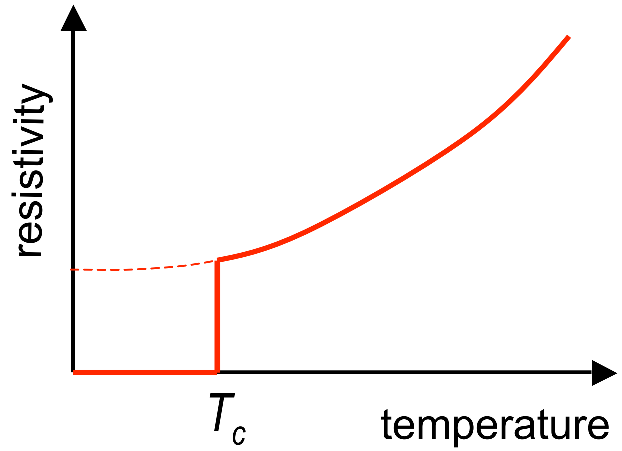
- symmetries of unconventional superconducting states
- cases study of tetragonal crystal within GL theory
- additionally broken symmetries and their physical properties
- symmetry based gap nodes
- case studies: Sr_2RuO_4, high-T_c superconductors
- complex phase diagrams
- non-centrosymmetric superconductors; non-unitary pairing

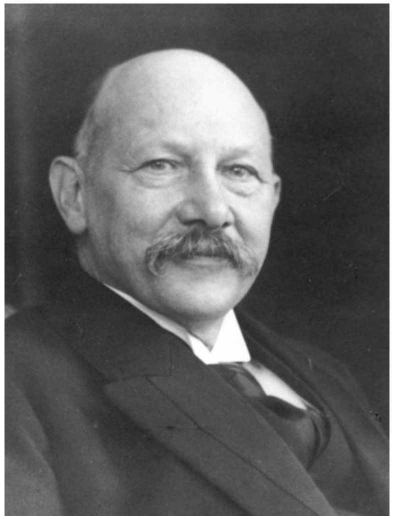
Superconductivity

Basics

Superconductivity

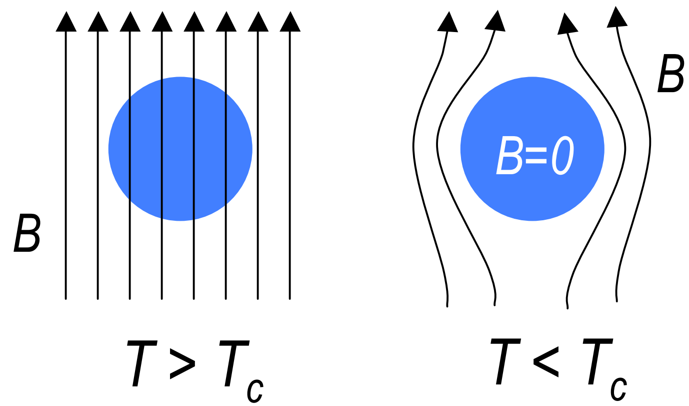
electrical resistance (1911)

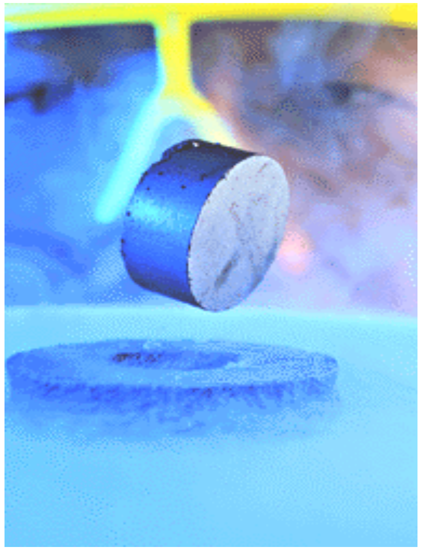




H. Kamerling-Onnes

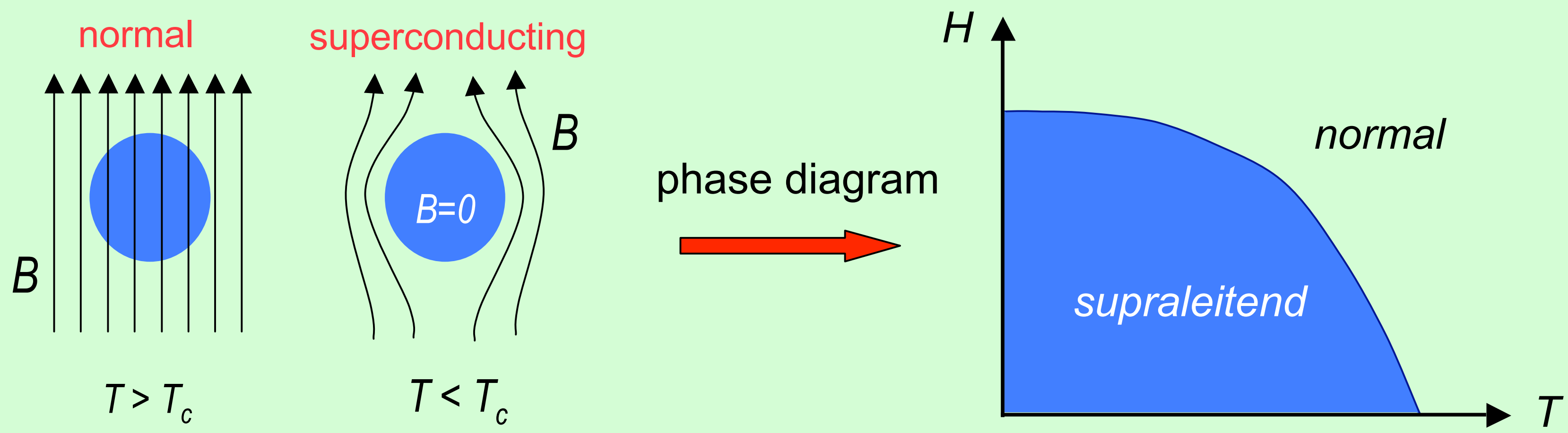
field expulsion (1933)
Meissner-Ochsenfeld effect





W. Meissner

Superconductivity as a thermodynamic phase



Thermodynamics

free enthalpy $dG = -SdT - \frac{1}{4\pi} B dH$

- normal

$$G_n(T,H) - G_n(T,0) = -\frac{H^2}{8\pi}$$

- superconducting

$$G_s(T,H) - G_s(T,0) = 0$$

phase boundary

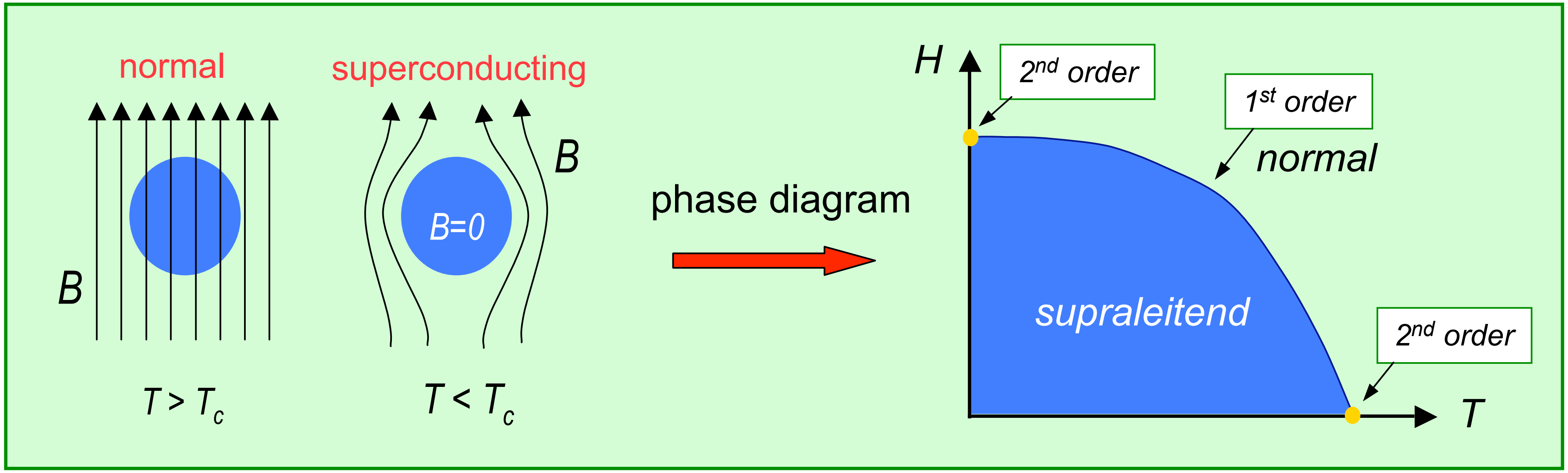
$$G_s(T,H_c) = G_n(T,H_c)$$

thermodynamic critical field $H_c(T)$

condensation energy

$$G_s(T,H) - G_n(T,H) = \frac{1}{8\pi}\left[H^2 - H_c^2(T)\right]$$

Superconductivity as a thermodynamic phase



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Screening effect - London theory (1935)

London Theory

- normal n_n and superfluid n_s electron density

$$n_s + n_n = n_e$$

- superfluid electrons: ideal incompressible fluid

velocity field: $\vec{v}_s(\vec{r}, t)$ with $\vec{\nabla} \cdot \vec{v}_s(\vec{r}, t) = 0$

dissipationless supercurrent: $\vec{j}(\vec{r}, t) = n_s e \vec{v}_s(\vec{r}, t)$

- vortex field $\vec{w}(\vec{r}, t) = \nabla \times \vec{v}_s(\vec{r}, t) + \frac{e}{mc}\vec{B}(\vec{r}, t)$

condition (postulate): $\vec{w}(\vec{r}, t) = 0$

everywhere and at all times inside the superconductor

Screening effect - London theory (1935)

London postulate

$$\vec{B} = -\frac{mc}{e}\nabla \times \vec{v}_s$$

Maxwell equation

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} = \frac{4\pi e n_s}{c}\vec{v}_s$$

$$\begin{aligned}\nabla \times (\nabla \times \vec{B}) &= -\nabla^2 \vec{B} \\ &= \frac{4\pi e n_s}{c}\nabla \times \vec{v}_s \\ &= -\frac{4\pi e^2 n_s}{mc^2}\vec{B}\end{aligned}$$

London equation

$$\nabla^2 \vec{B} = \lambda^{-2}\vec{B}$$

with

$$\lambda^{-2} = \frac{4\pi e^2 n_s}{mc^2}$$

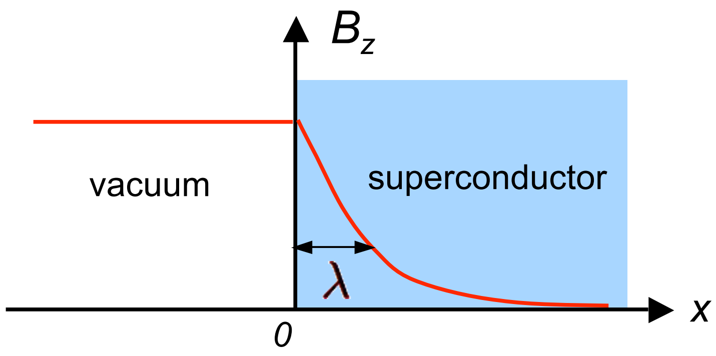
London penetration depth

Screening effect - London theory (1935)

Magnetic field penetration in superconductor

in superconductor $$\frac{d^2 B_z}{dx^2} = \lambda^{-2} B_z$$

$$B_z(x) = B_z(0) e^{-x/\lambda}$$



London equation

$$\nabla^2 \vec{B} = \lambda^{-2} \vec{B}$$ with $$\lambda^{-2} = \frac{4\pi e^2 n_s}{mc^2}$$ London penetration depth

Screening effect - London theory (1935)

Proca equation (massive photons)

$$A_\nu = (A_0, \vec{A})$$

$$\left\{\partial_\mu \partial^\mu + \left(\frac{\tilde{m}c}{\hbar}\right)^2\right\} A_\nu = 0$$

Lorentz gauge $\partial_\mu A^\mu = 0$

static and $A_0 = 0$

$$\vec{\nabla}^2 \vec{A} = \left(\frac{\tilde{m}c}{\hbar}\right)^2 \vec{A} = \lambda^{-2} \vec{A}$$

$$\nabla \cdot \vec{A} = 0$$

London equation

$$\nabla^2 \vec{B} = \lambda^{-2} \vec{B}$$

with $\lambda^{-2} = \dfrac{4\pi e^2 n_s}{mc^2}$ London penetration depth

$$\vec{B} = \nabla \times \vec{A}$$

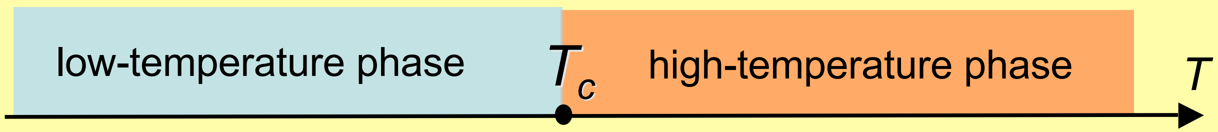
Phenomenological theory
of the Superconducting phase

Ginzburg-Landau Theory

Spontaneous Symmetry breaking

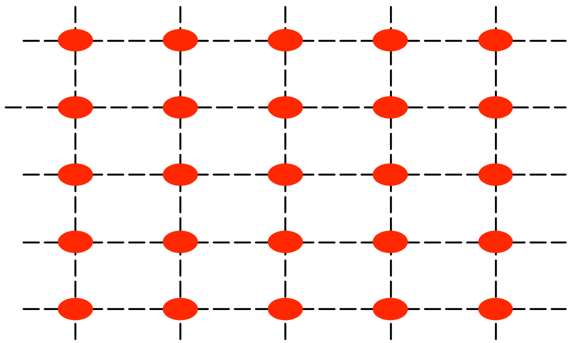
Landau theory of 2nd order phase transitions

concept of spontaneous symmetry breaking

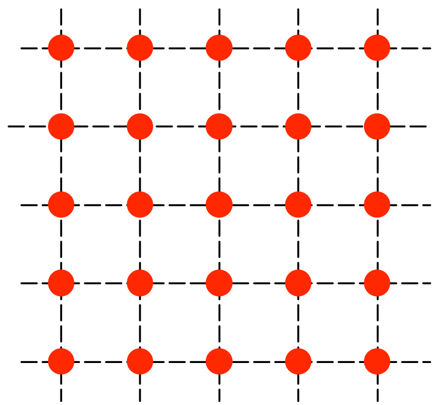


symmetry group $\mathcal{G}' \subset \mathcal{G}$

Example: structural transition



spontaneous lattice deformation



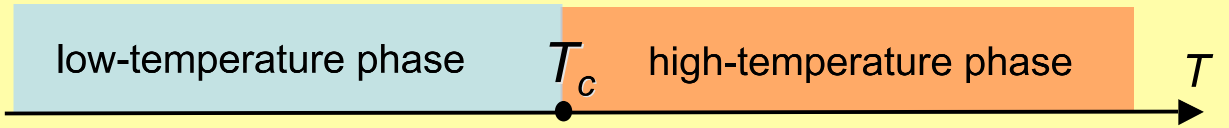
$$C_{2v} = \{E, C_2, \sigma_v, \sigma_v'\} \subset C_{4v} = \{E, C_4, C_4^3, C_2, \sigma_v, \sigma_v', \sigma_d, \sigma_d'\}$$

rectangular square

Spontaneous Symmetry breaking

Landau theory of 2nd order phase transitions

concept of order parameter

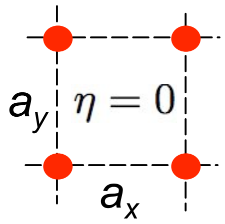


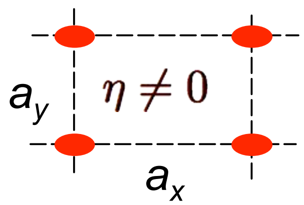
Order parameter $\eta \neq 0$ $\eta = 0$

Example: structural transition

natural order parameter

$$\eta = \frac{a_x - a_y}{a_x + a_y}$$



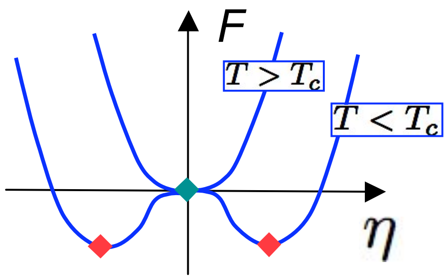


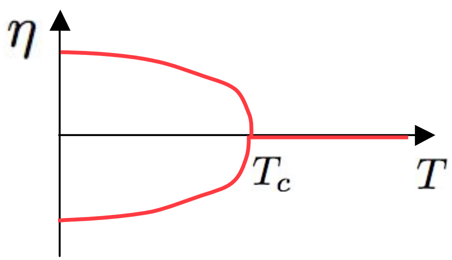
Landau free energy expansion

$$F(T;\eta) = a'(T - T_c)\eta^2 + b\eta^4$$

minimization

$$\eta^2 = \begin{cases} 0 & T > T_c \\ \frac{a'(T_c - T)}{2b} & T < T_c \end{cases}$$

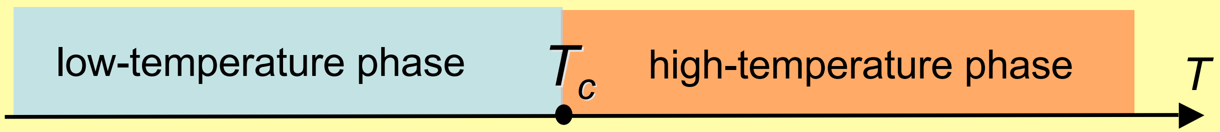




Spontaneous Symmetry breaking

Landau theory of 2nd order phase transitions

concept of order parameter



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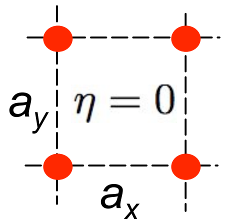
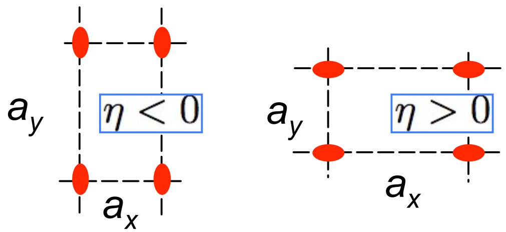


twin domains



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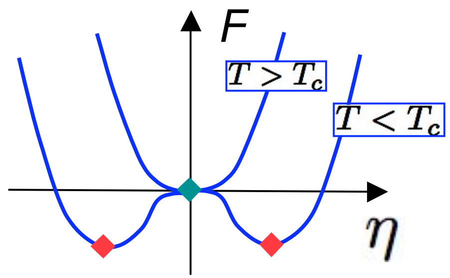

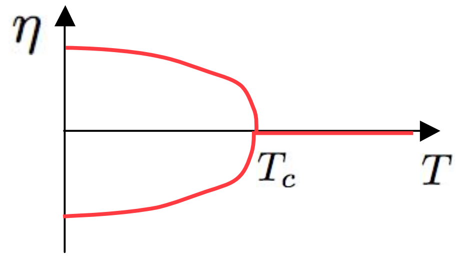

Spontaneous Symmetry breaking

construction of the Landau free energy:

$F(T;\eta)$ scalar under $\mathcal{G}$ invariants $\eta^2, \eta^4, \eta^6, \ldots$

$$\eta \xrightarrow{\{E, C_2, \sigma_v, \sigma_v'\}} +\eta$$

$$\eta \xrightarrow{\{C_4, C_4^3, \sigma_d, \sigma_d'\}} -\eta$$

$$\mathcal{G} = C_{4v}$$

$$F(T;\eta) = \underbrace{a'(T - T_c)}_{\text{sign change at } T_c}\eta^2 + b\eta^4 + \cdots$$

sign change at T_c

Example: structural transition

natural
order parameter

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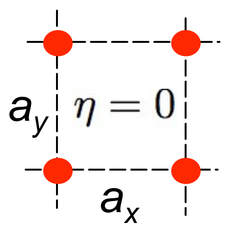

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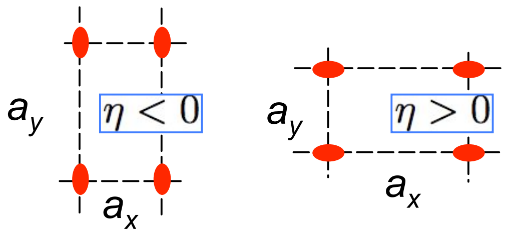

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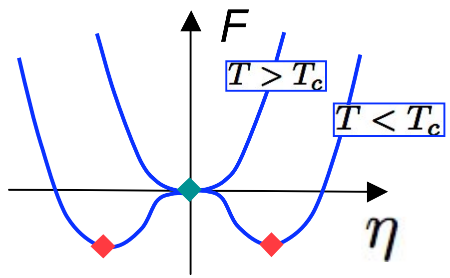

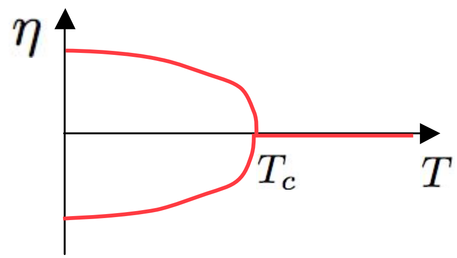

Superconductivity: spontaneously broken symmetry

Ginzburg-Landau theory (1950): 2nd order phase transition at T_c

order parameter:
macroscopic wave function

$$\psi(\vec{r}) = |\psi(\vec{r})| e^{i\phi(\vec{r})}$$

$$n_s = 2|\psi(\vec{r})|^2$$ density of superfluid electrons

symmetry aspects:

$\psi(\vec{r})$ scalar under (orbital/spin) rotations

vector potential

charged particles: $\psi(\vec{r}) \rightarrow \psi(\vec{r}) e^{i2e\chi(\vec{r})/\hbar c}$ $\quad \vec{A}(\vec{r}) \rightarrow \vec{A}(\vec{r}) + \vec{\nabla}\chi(\vec{r})$

U(1) gauge transformation

scalar free energy functional:

$$F[T;\psi,\vec{A}] = \int d^3r \left[a|\psi|^2 + b|\psi|^4 + \frac{1}{4m}\left| \left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A} \right) \psi \right|^2 + \frac{(\vec{\nabla} \times \vec{A})^2}{8\pi} \right]$$

Superconductivity: spontaneously broken symmetry

Invariance of gradient term: local *U(1)* gauge transformation

$$\psi = \psi' e^{-i2e\chi/\hbar c} \quad \text{and} \quad \vec{A} = \vec{A}' - \vec{\nabla}\chi$$

$$\begin{aligned}
\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right|^2 &= \left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}' + \frac{2e}{c}\vec{\nabla}\chi\right)\psi' e^{-i2e\chi/\hbar c}\right|^2 \\
&= \left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}' + \frac{2e}{c}\vec{\nabla}\chi - \frac{2e}{c}\vec{\nabla}\chi\right)\psi' e^{-i2e\chi/\hbar c}\right|^2 \\
&= \left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}'\right)\psi'\right|^2
\end{aligned}$$

$$F[T;\psi,\vec{A}] = \int d^3r \left[a|\psi|^2 + b|\psi|^4 + \frac{1}{4m}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right|^2 + \frac{(\vec{\nabla}\times\vec{A})^2}{8\pi}\right]$$

Superconductivity: spontaneously broken symmetry

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Invariance of gradient term: *time reversal K*

$$\psi \xrightarrow{\hat{K}} \psi' = \psi^* \qquad \text{and} \qquad \vec{A} \xrightarrow{\hat{K}} \vec{A}' = -\vec{A}$$

$$\begin{aligned}\left|\left(\frac{\hbar}{c}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right|^2 &= \left\{\left(-\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi^*\right\}\left\{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right\} \\ &= \left\{\left(-\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}'\right)\psi'\right\}\left\{\left(\frac{\hbar}{i}\vec{\nabla} + \frac{2e}{c}\vec{A}'\right)\psi'^*\right\} \\ &= \left|\left(\frac{\hbar}{c}\vec{\nabla} - \frac{2e}{c}\vec{A}'\right)\psi'\right|^2\end{aligned}$$

$$F[T;\psi,\vec{A}] = \int d^3r\left[a|\psi|^2 + b|\psi|^4 + \frac{1}{4m}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right|^2 + \frac{(\vec{\nabla}\times\vec{A})^2}{8\pi}\right]$$

Superconductivity: spontaneously broken symmetry

Ginzburg-Landau free energy functional

$$F[T;\psi,\vec{A}] = \int d^3r \left[a|\psi|^2 + b|\psi|^4 + \frac{1}{4m}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi\right|^2 + \frac{(\vec{\nabla}\times\vec{A})^2}{8\pi}\right]$$

equilibrium state: variational minimization with respect to ψ and $\vec{A}$

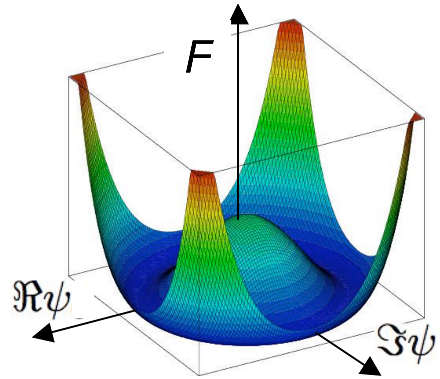
$$0 = \frac{\delta F}{\delta \psi^*} = a\psi + 2b\psi|\psi|^2 - \frac{1}{4m}\left\{\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right\}^2 \psi$$

$$0 = \frac{\delta F}{\delta \vec{A}} = \frac{1}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{e}{2mc}\left\{\psi^*\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi + \psi\left(-\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi^*\right\}$$

Order parameter - spontaneous symmetry breaking

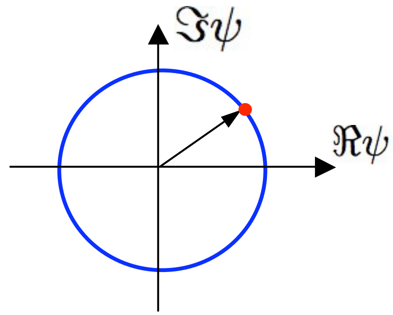
$$0 = \frac{\delta F}{\delta \psi^*} = a\psi + 2b\psi|\psi|^2 - \frac{1}{4m}\left\{\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right\}^2 \psi$$

uniform phase $\quad |\psi_0|^2 = -\frac{a}{2b} = \frac{a'(T_c - T)}{2b}$ $\quad T < T_c$



spontaneous symmetry breaking $\quad \psi_0 = |\psi_0| e^{i\phi}$

U(1) gauge symmetry



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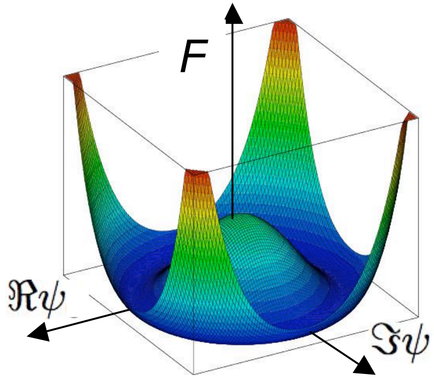
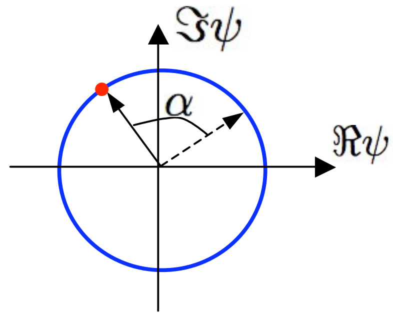
$T < T_c$



spontaneous symmetry breaking $\psi_0 = |\psi_0| e^{i\phi}$

U(1) gauge symmetry

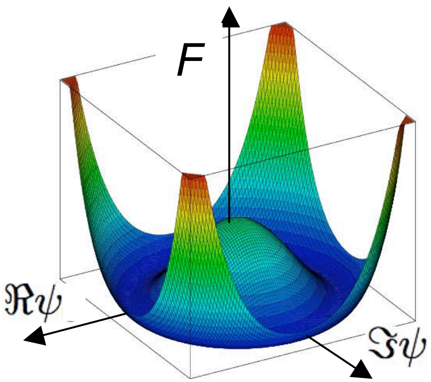
$$\psi_0 \to \psi_0 e^{i\alpha} \neq \psi_0$$



Order parameter - spontaneous symmetry breaking

$$0 = \frac{\delta F}{\delta \psi^*} = a\psi + 2b\psi|\psi|^2 - \frac{1}{4m}\left\{\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right\}^2 \psi$$

uniform phase $\quad |\psi_0|^2 = -\frac{a}{2b} = \frac{a'(T_c - T)}{2b}$ $\quad T < T_c$



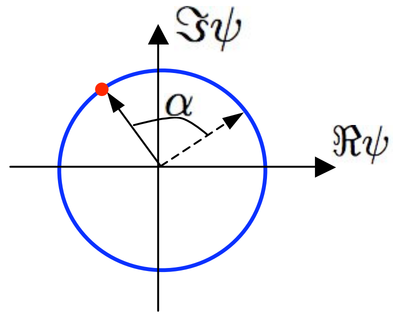
spontaneous symmetry breaking $\quad \psi_0 = |\psi_0| e^{i\phi}$

U(1) gauge symmetry

Stiffness: characteristic length scale

$$\psi - \xi^2 \left\{\vec{\nabla} - i\frac{2e}{\hbar c}\vec{A}\right\}^2 \psi - \psi\frac{|\psi|^2}{|\psi_0|^2} = 0$$

$$\psi_0 \to \psi_0 e^{i\alpha} \neq \psi_0$$



coherence length
(healing length)

$$\xi = \left\{\frac{\hbar^2}{4ma}\right\}^{1/2}$$

Vector potential - electromagnetic response

$$0 = \frac{\delta F}{\delta \vec{A}} = \frac{1}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{e}{2mc}\left\{\psi^*\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi + \psi\left(-\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi^*\right\}$$

Maxwell equation

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

supercurrent

$$\vec{j} = \frac{e\hbar}{2mi}\left\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right\} - \frac{2e^2}{mc}|\psi|^2\vec{A}$$

rigid order parameter (uniform)

$$\vec{\nabla}\times\vec{B} = \frac{8\pi e^2}{mc^2}|\psi_0|^2\vec{A} \quad \rightarrow \quad \vec{\nabla}^2\vec{B} = \frac{8\pi e^2}{mc^2}|\psi_0|^2\vec{B} = \lambda^{-2}\vec{B}$$

London equation

$$\lambda^{-2} = \frac{8\pi e^2}{mc^2}|\psi_0|^2 = \frac{4\pi e^2 n_s}{mc^2}$$

broken *U(1)*-gauge symmetry

$$\vec{\nabla}^2\vec{A} = \lambda^{-2}\vec{A}$$

massive photons

Anderson-Higgs mechanism

Vector potential - electromagnetic response

$$0 = \frac{\delta F}{\delta \vec{A}} = \frac{1}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \frac{e}{2mc}\left\{\psi^*\left(\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi + \psi\left(-\frac{\hbar}{i}\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\psi^*\right\}$$

Maxwell equation

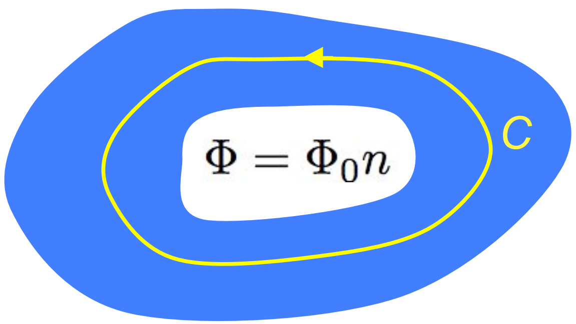
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

supercurrent

$$\vec{j} = \frac{e\hbar}{2mi}\left\{\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right\} - \frac{2e^2}{mc}|\psi|^2\vec{A}$$

$$\psi(\vec{r}) = |\psi_0|e^{i\phi(\vec{r})} \quad \rightarrow \quad \vec{j}(\vec{r}) = \frac{e\hbar}{m}|\psi_0|^2\left\{\vec{\nabla}\phi - \frac{2e}{\hbar c}\vec{A}\right\}$$

flux quantization



persistent current !

$$0 = \oint_C \vec{j}\cdot d\vec{s} = \frac{e\hbar}{m}|\psi_0|^2\oint_C\left\{\vec{\nabla}\phi - \frac{2e}{\hbar c}\vec{A}\right\}\cdot d\vec{s}$$

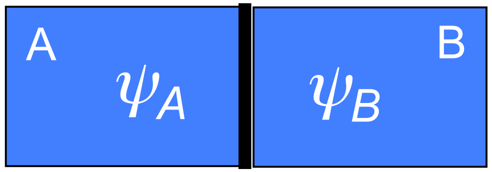
$$2\pi n = 2\pi\frac{\Phi}{\Phi_0}$$

flux quantum

$$\Phi_0 = \frac{hc}{2e}$$

Macroscopic quantum coherence

Josephson effect (1962): tunneling between two superconductors



$\psi_A = |\psi_0| e^{i\phi_A} \quad \psi_B = |\psi_0| e^{i\phi_B}$

$$\vec{j} = \frac{e\hbar}{2mi} \left\{ \psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right\}$$

discrete current expression (*d* effective distance)

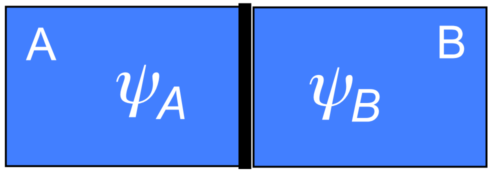
$$j = \frac{e\hbar}{2mid} \{ (\psi_A^* + \psi_B^*)(\psi_A - \psi_B) - (\psi_A^* - \psi_B^*)(\psi_A + \psi_B) \}$$

$$j = \frac{e\hbar}{mid} \{ \psi_B^* \psi_A - \psi_A^* \psi_B \} = \underbrace{\frac{2e\hbar}{md} |\psi_0|^2}_{j_c} \sin(\phi_A - \phi_B)$$

j_c Josephson critical current

Macroscopic quantum coherence

Josephson effect (1962): tunneling between two superconductors

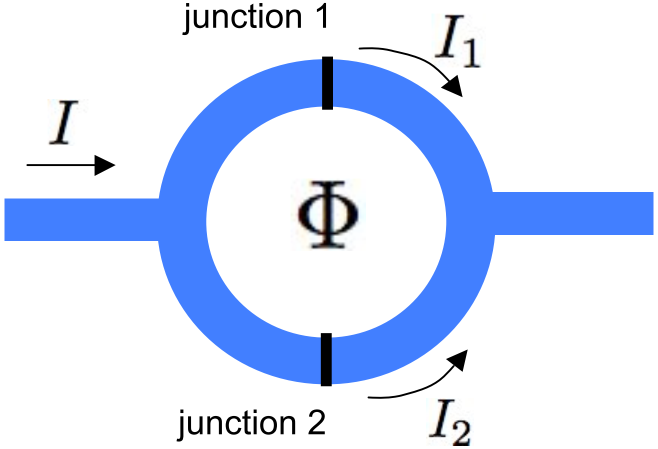


$\psi_A = |\psi_0| e^{i\phi_A}$ $\psi_B = |\psi_0| e^{i\phi_B}$

$$J = J_c \sin(\phi_A - \phi_B)$$

Aharonov-Bohm effect: *SQUID*

$\Delta\phi_1 = \phi - \pi\Phi/\Phi_0$

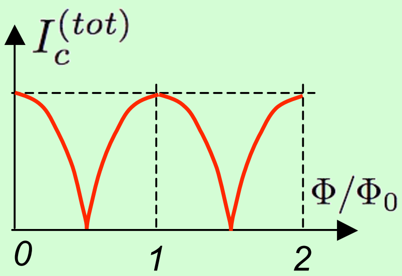


$\Delta\phi_2 = \phi + \pi\Phi/\Phi_0$

$$I_{c1} = I_{c2} = I_c$$

$$I = I_1 + I_2 = I_c \{\sin \Delta\phi_1 + \sin \Delta\phi_2\}$$

$$I = 2I_c \sin\phi \cos \pi\Phi/\Phi_0$$



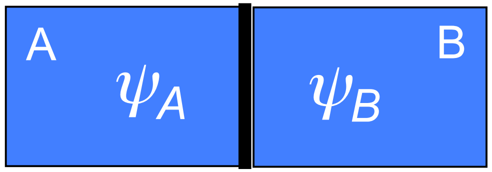
Interference pattern

Superconducting QUantum Interference Device

$$I_c^{(tot)}(\Phi) = 2I_c \left|\cos \pi\Phi/\Phi_0\right|$$

Macroscopic quantum coherence

Josephson effect (1962): tunneling between two superconductors



$$J = J_c \sin(\phi_A - \phi_B)$$

$\psi_A = |\psi_0| e^{i\phi_A}$ $\psi_B = |\psi_0| e^{i\phi_B}$

„Bernoulli“ equation (London)

$$\frac{\partial \vec{v}_s}{\partial t} = -\vec{\nabla} \frac{\vec{v}_s^2}{2} + \frac{e}{m} \vec{E} \quad \rightarrow \quad \frac{\partial \phi}{\partial t} = -\frac{mc}{\hbar n_s^2 e^2} \vec{j}_s^2 - \frac{2\pi c}{\Phi_0} A_0$$

voltage drop over a junction

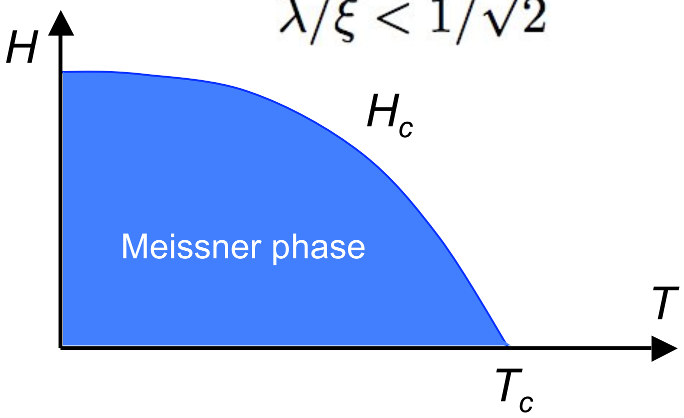
ac-Josephson effect

$$\frac{\partial}{\partial t}(\phi_A - \phi_B) = \omega = \frac{2e}{\hbar} V \quad \rightarrow \quad J(t) = J_c \sin(\omega t)$$

Vortices and vortex lattice

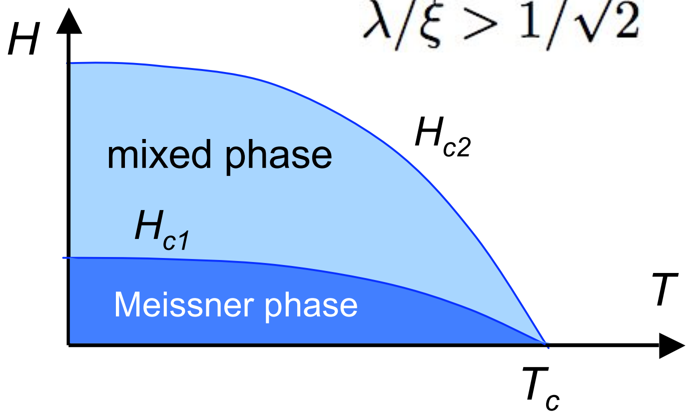
type I superconductor

$\lambda/\xi < 1/\sqrt{2}$



type II superconductor

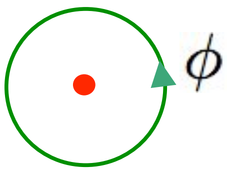
$\lambda/\xi > 1/\sqrt{2}$

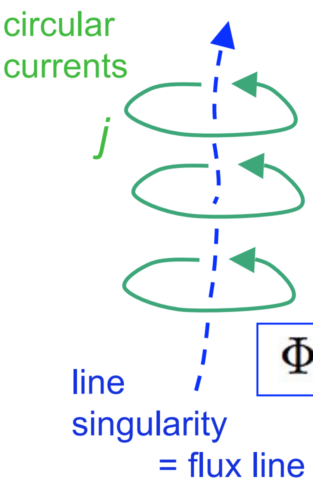


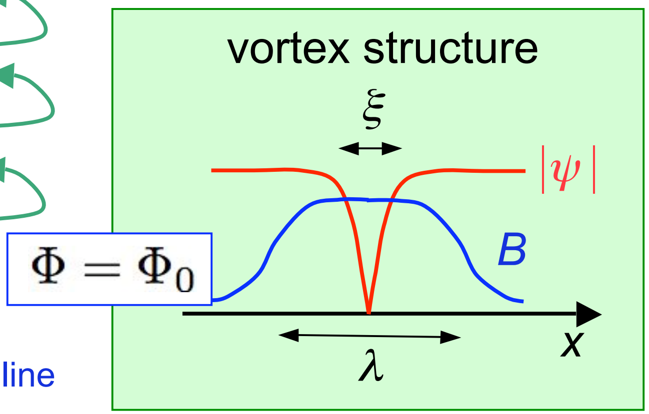
Mixed phase: Abrikosov vortex lattice (1957)

vortex: line singularity with phase winding

$$\psi(\vec{r}) = |\psi_0| e^{i\phi(\vec{r})}$$







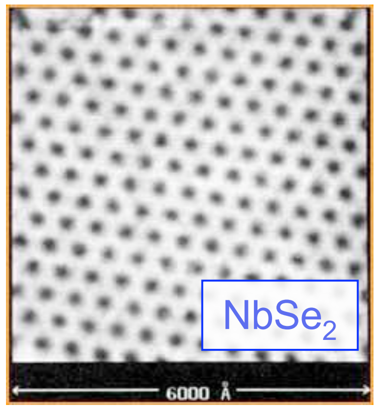
repulsive flux lines form lattice

broken translational symmetry

stable lattice
triangular

image
Scanning
Tunneling
Microscope

Hess et al. (1989)



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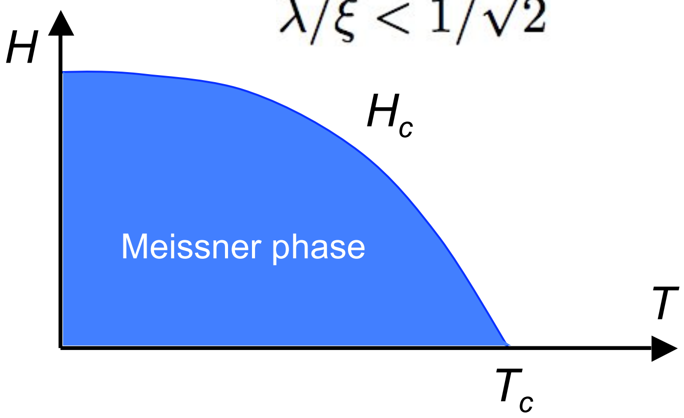

type II superconductor

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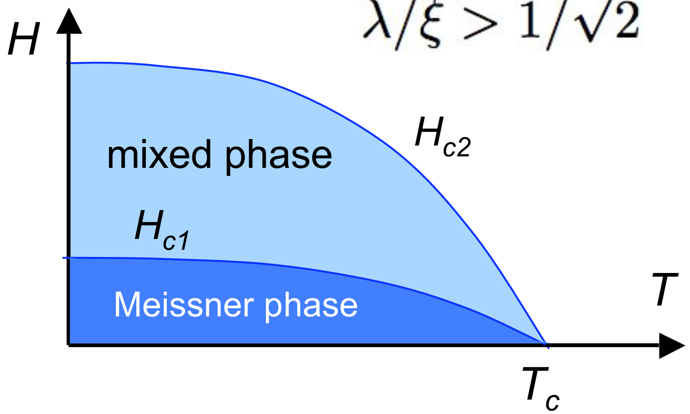

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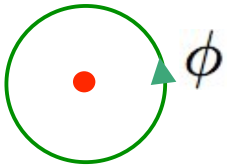
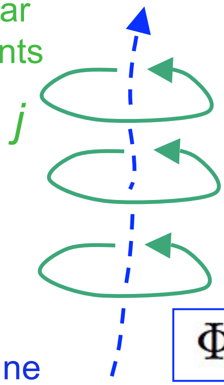
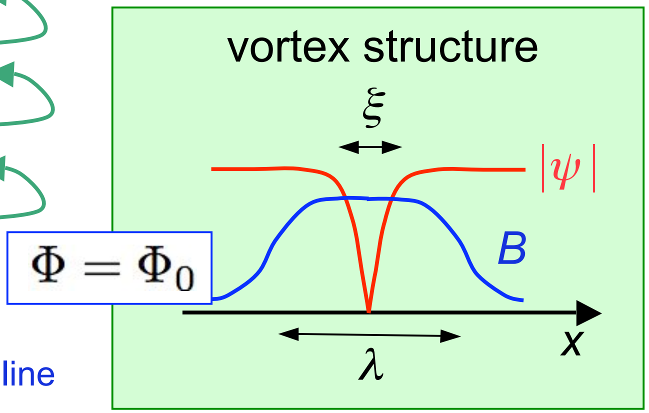






repulsive flux lines form lattice

broken translational symmetry

stable lattice

triar

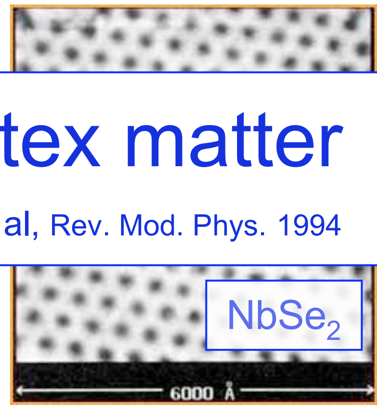
image

Scanni

Tunneling

Microscope

Hess et al. (1989)

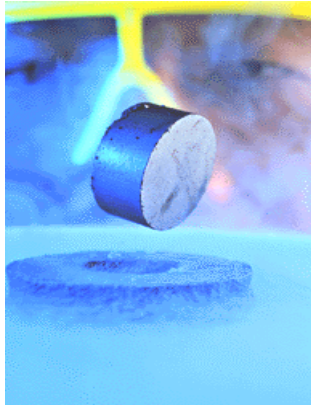


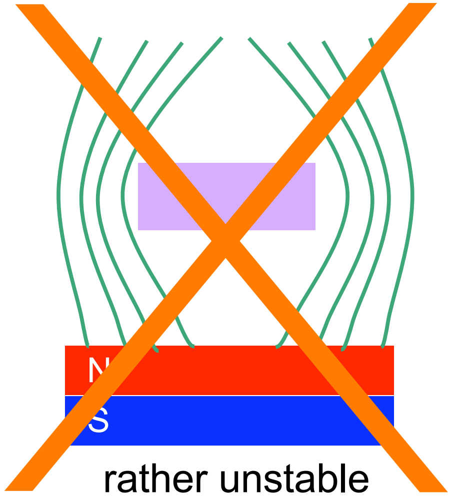
Vortex matter

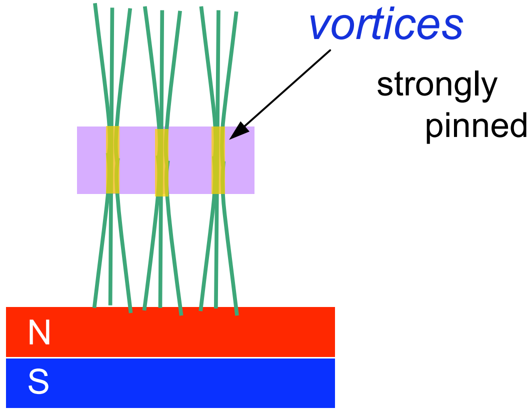
Blatter et al, Rev. Mod. Phys. 1994

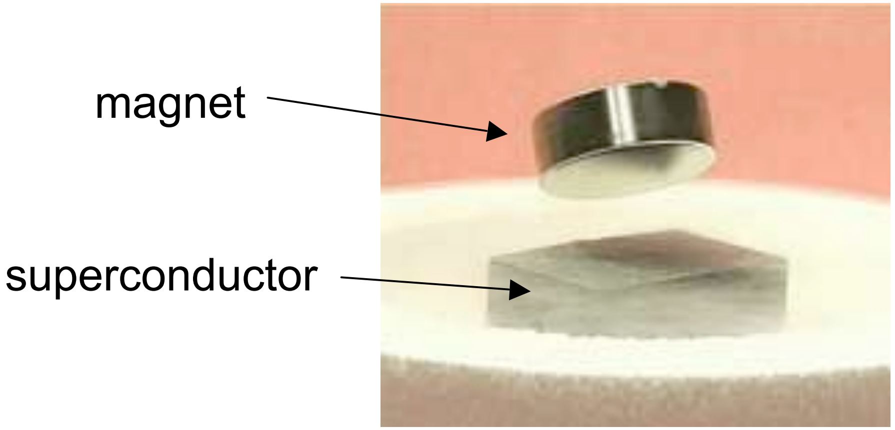
Vortices and vortex lattice

Levitation not due to Meissner-Ochsenfeld effect!





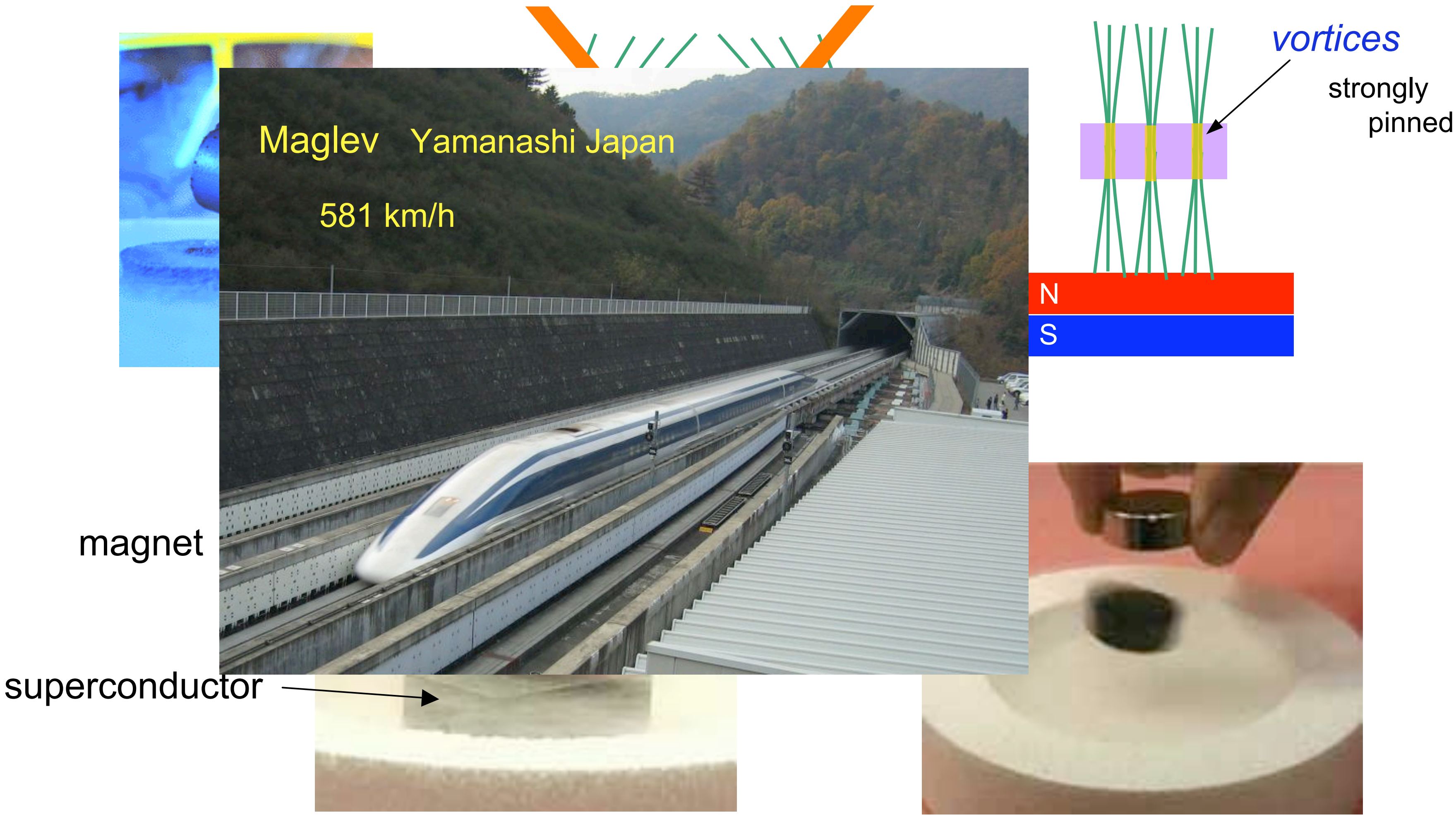




SUPERCONDUCTOR LAB OSLO

Vortices and vortex lattice

Levitation not due to Meissner-Ochsenfeld effect!



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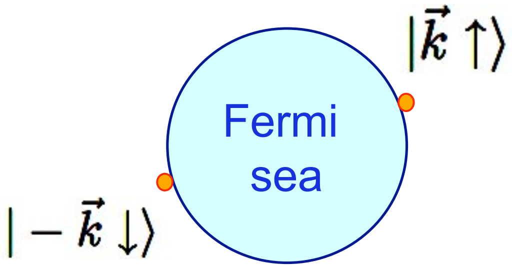
Bardeen-Cooper-Schrieffer

Microscopic theory of superconductivity

Mircroscopic basis for GL order parameter

Bardeen-**C**ooper-**S**chrieffer (1957) microscopic understanding



formation of *Cooper pairs* due to attractive interaction

$|\vec{k}\uparrow\rangle$, $|-\vec{k}\downarrow\rangle$ ⟹ vanishing
- total momentum $P_{tot} = 0$
- rel. angular momentum $L_{rel} = 0$
- total spin, spin singlet $S_{tot} = 0$

Many-body state: *coherent state* of Cooper pairs

BCS-variational state: $$|\Psi\rangle = \prod_{|\vec{k}|\leq k_F} \left\{ u_{\vec{k}} + v_{\vec{k}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} \right\} |vac\rangle$$

pair wave function $$\Psi_{\vec{k}} = \langle \underset{N}{\Psi} | c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} | \underset{N+2}{\Psi} \rangle = u_{\vec{k}} v_{\vec{k}}$$ „order parameter“

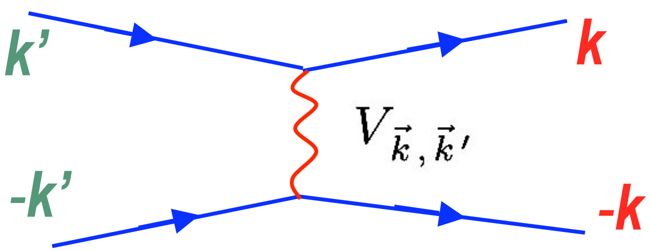
U(1) gauge transformation $$c_{\vec{k}s} \rightarrow c_{\vec{k}s} e^{ie\chi/\hbar c} \quad \Rightarrow \quad \Psi_{\vec{k}} \rightarrow \Psi_{\vec{k}} e^{i2e\chi/\hbar c}$$

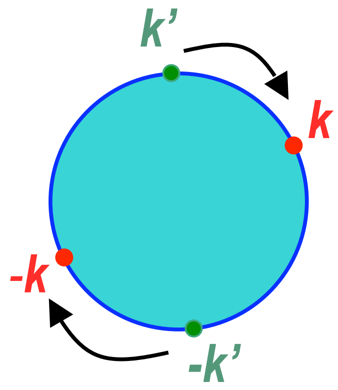
$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi$$

Pairing interaction

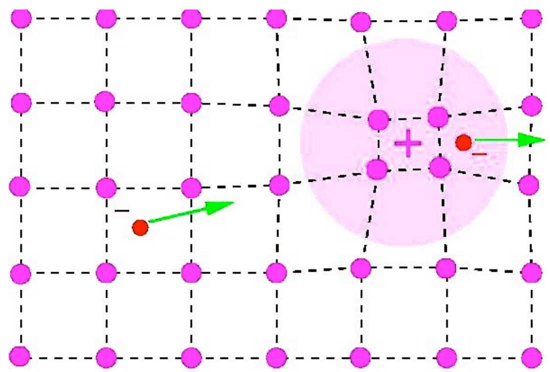
Cooper pair formation (bound state of 2 electrons) needs attractive interaction

$$\mathcal{H}_{pair} = \frac{1}{2\Omega} \sum_{\vec{k},\vec{k}'} \sum_{s,s'} V_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k},s} c^{\dagger}_{-\vec{k},s'} c_{-\vec{k}',s'} c_{\vec{k}',s}$$





electron phonon interaction:



electrons polarize their environment

renormalized Coulomb interaction

$$V_{\vec{k},\vec{k}'} = \frac{4\pi e^2}{q^2} \quad \Rightarrow \quad V_{\vec{k},\vec{k}'} = \frac{4\pi e^2}{q^2 \varepsilon(\vec{q},\omega)}$$

BCS mean field theory

simple model:

$$\mathcal{H} = \underbrace{\sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s}}_{\text{band energy}} + \underbrace{g \sum_{\vec{k},\vec{k}'} c^{\dagger}_{\vec{k}\uparrow} c^{\dagger}_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}}_{\text{pairing interaction}}$$

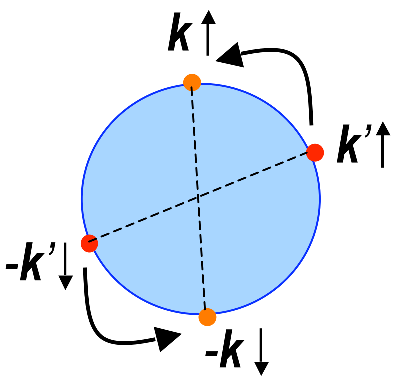
band energy:

$$\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu = \frac{\hbar^2}{2m}(\vec{k}^{\,2} - k_F^2)$$

pairing interaction:

$$U(\vec{r} - \vec{r}) = g\delta^{(3)}(\vec{r} - \vec{r}\,')$$

attractive contact Interaction $g < 0$



$$V(\vec{q} = \vec{k} - \vec{k}\,') = \int d^3r\, U(\vec{r}) e^{i\vec{q}\cdot\vec{r}} = g = V_{\vec{k},\vec{k}'}$$

consider only scattering between zero-momentum electron pairs of opposite spin (spin singlet)

BCS mean field theory

simple model: $$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}$$

many-body problem treated by mean field theory:

decoupling of interaction term

mean fields: $$\rho_{\vec{q}} = \sum_{\vec{k},s} \langle c^\dagger_{\vec{k}+\vec{q}s} c_{\vec{k}s} \rangle$$ particle density (Hartree-Fock)

$$\vec{S}_{\vec{q}} = \sum_{\vec{k}} \sum_{s,s'} \langle c^\dagger_{\vec{k}s} \vec{\sigma}_{ss'} c_{\vec{k}s'} \rangle$$ spin density (Hartree-Fock)

→ $$b_{\vec{k}} = \langle c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} \rangle$$ BCS - “off diagonal”

BCS mean field theory

simple model:

$$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow}$$

replace

$$c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} = b^*_{\vec{k}} + \{c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}}\}\,, \quad c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} = b_{\vec{k}} + \{c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} - b_{\vec{k}}\}$$

mean field Hamiltonian:

$$\begin{aligned}\mathcal{H}_{mf} &= \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} \{b^*_{\vec{k}'} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + b_{\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}} b_{\vec{k}'}\} \\ &= \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \sum_{\vec{k}} \left\{\Delta^* c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - \Delta^* b_{\vec{k}}\right\}\end{aligned}$$

with

$$\Delta^* = -g \sum_{\vec{k}'} b^*_{\vec{k}'}\,, \qquad \Delta = -g \sum_{\vec{k}'} b_{\vec{k}'}$$

BCS mean field theory

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + g \sum_{\vec{k},\vec{k}'} \{ b^*_{\vec{k}'} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + b_{\vec{k}'} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - b^*_{\vec{k}} b_{\vec{k}'} \}$$

$$= \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \sum_{\vec{k}} \left\{ \Delta^* c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} + \Delta c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} - \Delta^* b_{\vec{k}} \right\}$$

find quasiparticle states with $\frac{\partial}{\partial t} \gamma^\dagger_{\vec{k}} = i[\mathcal{H}_{mf}, \gamma^\dagger_{\vec{k}}] = E_{\vec{k}} \gamma^\dagger_{\vec{k}}$

Bogolyubov-transformation

$$c_{\vec{k}\uparrow} = u^*_{\vec{k}} \gamma_{\vec{k}1} + v_{\vec{k}} \gamma^\dagger_{\vec{k}2}$$
$$c^\dagger_{-\vec{k}\downarrow} = -v^*_{\vec{k}} \gamma_{\vec{k}1} + u_{\vec{k}} \gamma^\dagger_{\vec{k}2}$$

$$|u_{\vec{k}}|^2 + |v_{\vec{k}}|^2 = 1$$

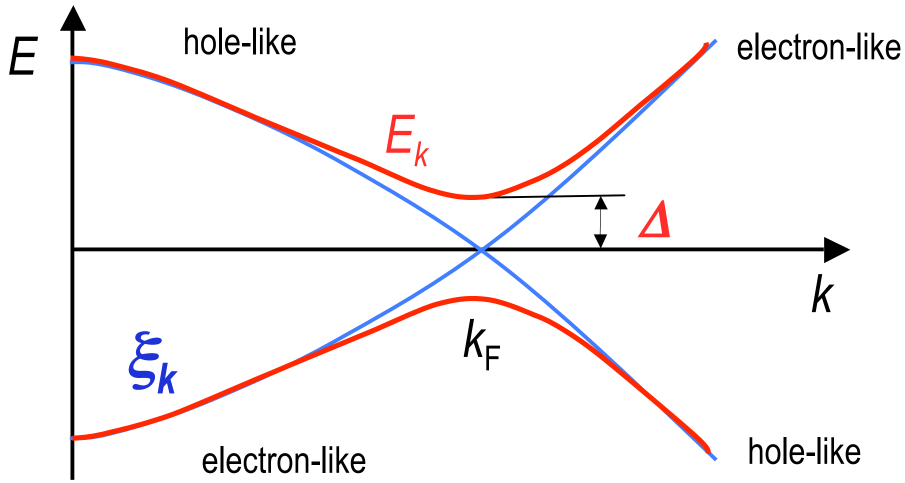
➡ quasiparticle energy $E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + \Delta^2}$

➡ $$\mathcal{H} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] + \sum_{\vec{k}} E_{\vec{k}} (\gamma^\dagger_{\vec{k}1} \gamma_{\vec{k}1} + \gamma^\dagger_{\vec{k}2} \gamma_{\vec{k}2})$$

Quasiparticle Spectrum

$$\mathcal{H} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] + \sum_{\vec{k}} E_{\vec{k}} (\gamma^\dagger_{\vec{k}1}\gamma_{\vec{k}1} + \gamma^\dagger_{\vec{k}2}\gamma_{\vec{k}2})$$

$$E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + \Delta^2}$$



quasiparticle excitation gap: Δ

condensation energy gain due to gap

Self-consistence equation:

$$\Delta = -g \sum_{\vec{k}} b_{\vec{k}} = -g \sum_{\vec{k}} u^*_{\vec{k}} v_{\vec{k}} [1 - f(E_{\vec{k}})]$$

$$= -g \sum_{\vec{k}} \frac{\Delta}{2E_{\vec{k}}} \tanh\left(\frac{E_{\vec{k}}}{k_B T}\right)$$

Fermi distribution function

$$f(E) = \frac{1}{1 + e^{E/k_B T}}$$

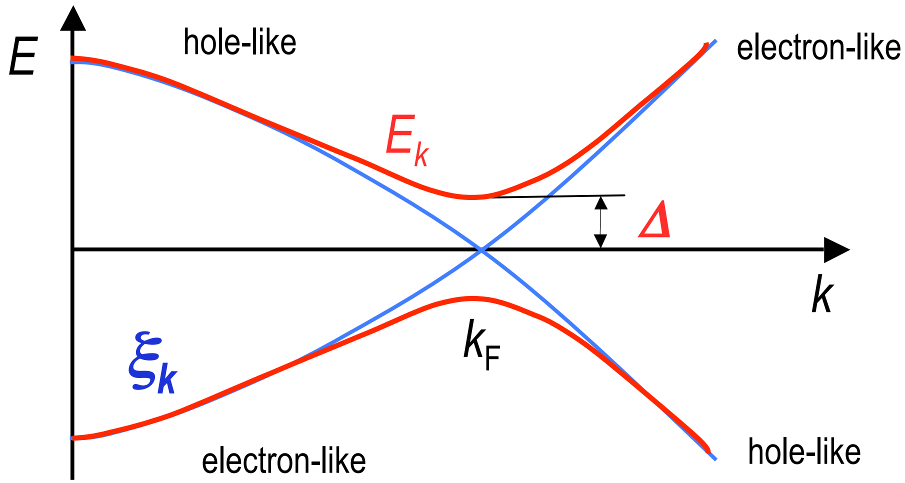
solution only for $g < 0$ attractive

Quasiparticle Spectrum

$$\mathcal{H} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] + \sum_{\vec{k}} E_{\vec{k}} (\gamma^{\dagger}_{\vec{k}1}\gamma_{\vec{k}1} + \gamma^{\dagger}_{\vec{k}2}\gamma_{\vec{k}2})$$

$$E_{\vec{k}} = \sqrt{\xi^2_{\vec{k}} + \Delta^2}$$

quasiparticle excitation gap: Δ



condensation energy at T=0:

$E_{cond} = E_s - E_n$

energy gain due to gap

$$E_{cond} = \sum_{\vec{k}} [\xi_{\vec{k}} - E_{\vec{k}} + \Delta b_{\vec{k}}] = -\frac{1}{2} N(0) |\Delta|^2$$

weak-coupling regime

depends on density of states at the Fermi surface and the gap magnitude

Summary

spontaneous symmetry breaking
order parameter
macroscopic wave function

$$\psi(\vec{r}) \quad \text{with} \quad |\psi(\vec{r})|^2 = n_s/2$$

violating *U(1)* gauge symmetry

Cooper pairing of electrons

$|\vec{k}\uparrow\rangle$

$|-\vec{k}\downarrow\rangle$

$$\Psi_{\vec{k}} = \langle\Psi|c_{-\vec{k}\downarrow}c_{\vec{k}\uparrow}|\Psi\rangle$$

quasiparticle gap

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2}$$

Bardeen-Cooper-Schrieffer

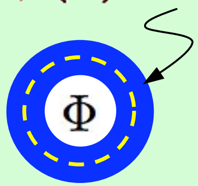
Anderson-Higgs mechanism

$$\vec{\nabla}\vec{A} = \lambda^{-2}\vec{A} = \frac{4\pi e^2 n_s}{mc^2}\vec{A}$$

London equation
Meissner-Ochsenfeld screening

Flux quantization

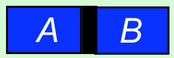
$$\psi(\vec{r}) = |\psi(\vec{r})|e^{i\phi(\vec{r})}$$



$$\Phi = \Phi_0 n = \frac{hc}{2e}n$$

persistent current

Macroscopic coherence



$$J = J_c \sin(\phi_B - \phi_A)$$

$$\frac{\partial}{\partial t}(\phi_B - \phi_A) = \frac{2e}{\hbar}V$$

Josephson effect

Generalization of BCS theory

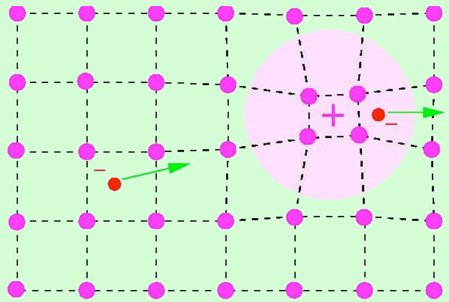
Mechanisms of pairing

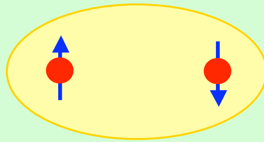
conventional Cooper pairing

BCS theory

attractive interaction by
electron-phonon interaction

"contact interaction"





angular momentum
l =0
spin singlet

unconventional Cooper pairing

strongly electron correlation: dominating Coulomb repulsion ("contact interaction")

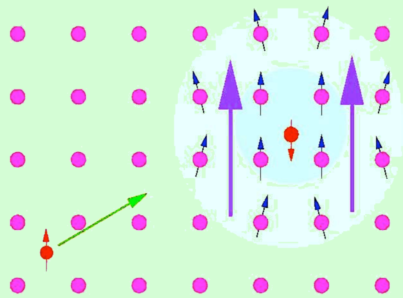
➡ *l =0* channel handicaped ➡ *l >0* avoiding Coulomb repulsion

longer-ranged interactions

Kohn & Luttinger: pairing due to repulsive interaction
(Friedel oscillations)

Berk & Schrieffer: pairing mediated by spin fluctuations

spin polarizable near
magnetic instability



. . . .

Magnetic mechanism !?

High-temperature superconductors

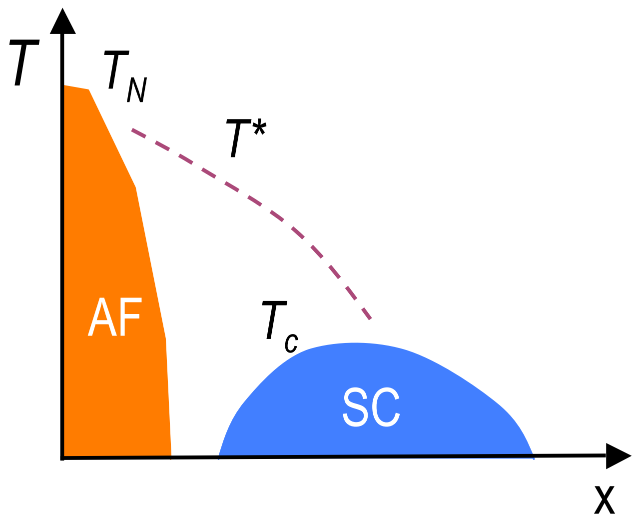
Layered perovskite copper-oxides

Müller & Bednorz (1986)

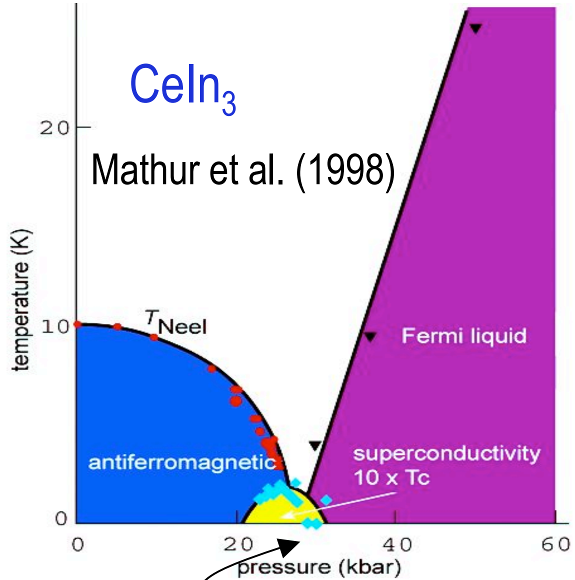
$La_{2-x}Sr_xCuO_4$ T_c=45K

$YBa_2Cu_3O_{6+\delta}$ T_c=92K

$HgBa_2Ca_2Cu_3O_9$ T_c=133.5K



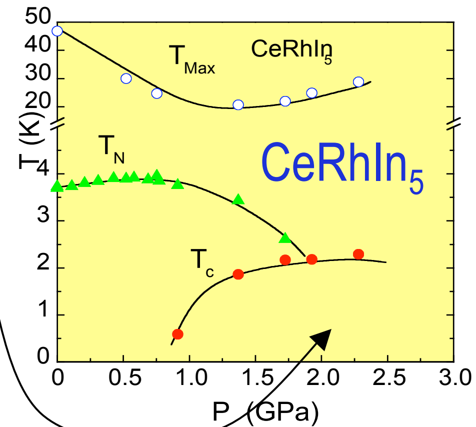
Heavy Fermion superconductors:



Quantum Critical point

AF ⟷ PM

Thompson et al. (2001)



Generalization of Cooper pairing

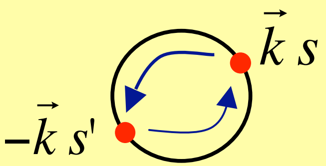
generalized BCS coherent state:

$$|\Psi\rangle = \prod_{|\vec{k}|\leq k_F} \sum_{s,s'} \left\{ u_{\vec{k}ss'} + v_{\vec{k}ss'} c^{\dagger}_{\vec{k}s} c_{-\vec{k}s'} \right\} |vac\rangle$$

pair wave function: $$\Psi_{\vec{k}ss'} = \langle \Psi | c_{-\vec{k}s'} c_{\vec{k}s} | \Psi \rangle$$

Symmetry of pairs of identical electrons: $$\Psi_{\vec{k}ss'} = \underbrace{g(\vec{k})}_{\text{orbital}} \; \underbrace{\chi_{ss'}}_{\text{spin}}$$

wave function totally antisymmetric under particle exchange



$$\vec{P}_{tot} = 0$$

$$\vec{k} \rightarrow -\vec{k} \qquad s \leftrightarrow s'$$

even parity — L =0,2,4,… , S=0 — singlet
(L even, S odd)

odd parity — L = 1,3,5,… , S=1 — triplet
(L odd, S even)

Key symmetries

Anderson's Theorems (1959,1984)

Cooper pairs with total momentum $P_{tot}=0$ form from degenerate quasiparticle states. *How to guarantee degenerate partners?*

- **Spin singlet pairing:** time reversal symmetry

$$|\vec{k}\uparrow\rangle \rightarrow \hat{T}|\vec{k}\uparrow\rangle = |-\vec{k}\downarrow\rangle \quad \Longleftrightarrow \quad \epsilon_{\vec{k}\uparrow} = \epsilon_{-\vec{k}\downarrow}$$

harmful: magnetic impurities, ferromagnetism, Zeemann fields (paramagnetic limiting)

- **Spin triplet pairing:** time reversal & inversion symmetry

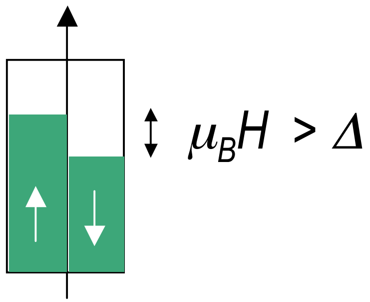
$$|\vec{k}\uparrow\rangle \rightarrow \begin{cases} \hat{I}|\vec{k}\uparrow\rangle = |-\vec{k}\uparrow\rangle \\ \hat{T}|\vec{k}\uparrow\rangle = |-\vec{k}\downarrow\rangle \\ \hat{T}\hat{I}|\vec{k}\uparrow\rangle = |\vec{k}\downarrow\rangle \end{cases} \quad \Longleftrightarrow \quad \begin{aligned} \epsilon_{\vec{k}\uparrow} &= \epsilon_{-\vec{k}\uparrow} \\ &= \epsilon_{-\vec{k}\downarrow} \\ &= \epsilon_{\vec{k}\downarrow} \end{aligned}$$

harmful: crystal structure without inversion center

Paramagnetic limiting: *lack of time reversal symmetry*

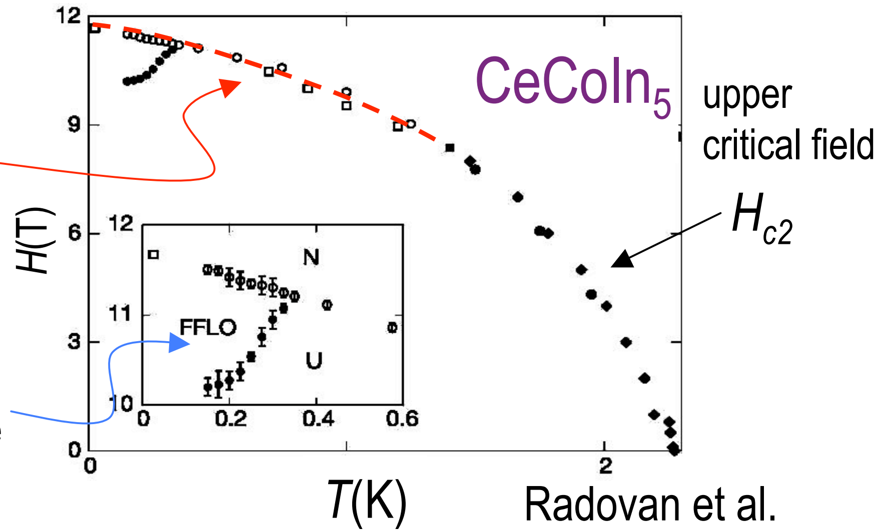
Zeeman splitting of Fermi surfaces exceeds the gap magnitude

→ No singlet pairing possible



Paramagnetic suppression
1^{st} order transition

modulated Fulde-Ferrel-Larkin-Ovchinikov phase



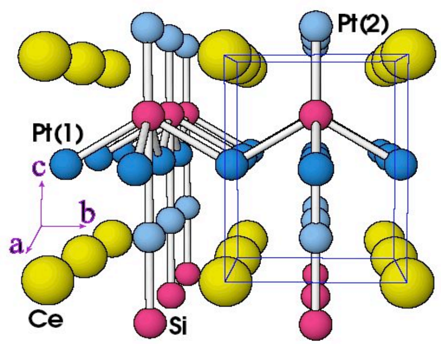
Radovan et al.

Antisymmetric spin-orbit coupling: *lack of inversion symmetry*

Crystal structure without an inversion center

e.g. $CePt_3Si$

no mirror plane for $z \longrightarrow -z$



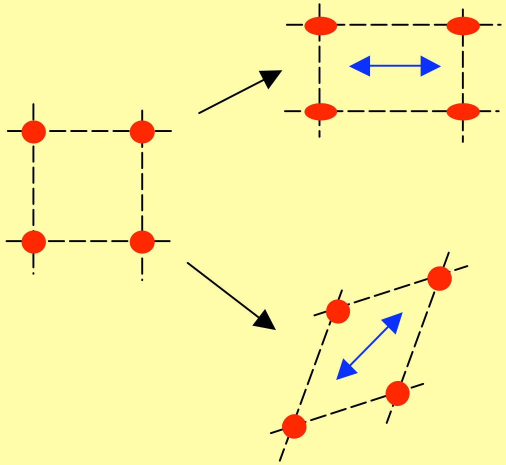
Bauer et al.

Generalized Cooper pairing

Symmetry aspects for the pair wave function $\Psi_{\vec{k}ss'} = \langle \Psi | c_{-\vec{k}s'} c_{\vec{k}s} | \Psi \rangle$

➡ many different order parameters of superconductivity

Example lattice distortion:



deformation - strain

$\epsilon_{B_1} = \epsilon_{xx} - \epsilon_{yy}$

$\epsilon_{B_2} = \epsilon_{xy}$

Classification by irreducible representations

point group C_{4v}

Γ	$\epsilon_{\mu\nu}$
A_1	$\epsilon_{xx} + \epsilon_{yy}$
A_2	none
B_1	$\epsilon_{xx} - \epsilon_{yy}$
B_2	ϵ_{xy}

strain $\epsilon_{\mu\nu} = \frac{1}{2}\left(\frac{\partial u_\mu}{\partial r_\nu} + \frac{\partial u_\nu}{\partial r_\mu}\right)$

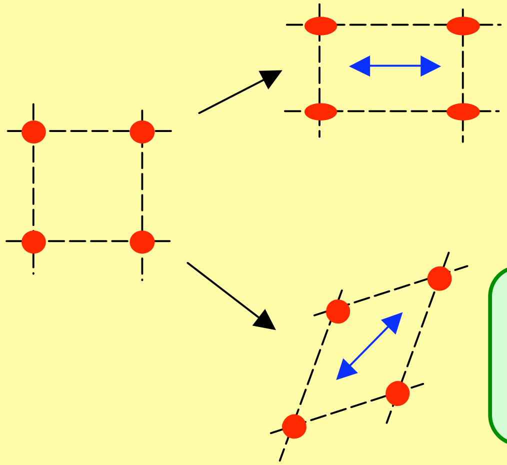
displacement field $\vec{u}(\vec{r})$

Generalized Cooper pairing

Symmetry aspects for the pair wave function $\Psi_{\vec{k}ss'} = \langle \Psi | c_{-\vec{k}s'} c_{\vec{k}s} | \Psi \rangle$

→ many different order parameters of superconductivity

Example lattice distortion:



deformation - strain

$$\epsilon_{B_1} = \epsilon_{xx} - \epsilon_{yy}$$

specific critical temperature for each representation

Classification by irreducible representations

point group C_{4v}

Γ	$\epsilon_{\mu\nu}$
A_1	$\epsilon_{xx} + \epsilon_{yy}$
A_2	none
B_1	$\epsilon_{xx} - \epsilon_{yy}$
B_2	ϵ_{xy}

order parameter

$$\epsilon_{\mu\nu} = \sum_{\Gamma_n} \eta_{\Gamma_n} \epsilon_{\Gamma_n}$$

Landau free energy

$$\rightarrow \quad F = \sum_{\Gamma_n} \{ a'_n (T - T_c(\Gamma_n)) \eta^2_{\Gamma_n} + b_n \eta^4_{\Gamma_n} \} + \cdots$$

Generalized Ginzburg-Landau theory

Landau's recipe: order parameters belong to irreducible representations of the normal state symmetry group

$$\psi = \sum_{\gamma \in \Gamma} \eta_\gamma \psi_\gamma$$

$\{\psi_1, \psi_2, \ldots\}$ basis set of irreducible representation Γ

η_γ transform according to Γ

formulate a free energy functional as a scalar function of η_γ

$$F[\eta_m] = \int d^3r [a \sum_m |\eta_m|^2 + \sum_{m_1,\ldots m_4} b_{m_1,\ldots m_4} \eta^*_{m_1} \eta^*_{m_2} \eta_{m_3} \eta_{m_4}$$

$$+ \sum_{m_1,m_2} \sum_{n_1,n_2} K_{m_1 m_2 n_1 n_2} \left(D_{n_1} \eta_{m_1}\right)^* \left(D_{n_2} \eta_{m_2}\right) + \frac{1}{8\pi} \left(\vec{\nabla} \times \vec{A}\right)^2]$$

invariant under all symmetry operations of rotations, time reversal and *U(1)*-gauge

$a = a'(T - T_c)$, $b_m, K_{mm'nn'}$ real constants, $\vec{D} = \vec{\nabla} - i \frac{2e}{\hbar c} \vec{A}$ gauge-invariant gradient

Generalized Cooper pairing

Parametrization of pair wave function $\Psi_{\vec{k}ss'} = \langle \Psi | c_{-\vec{k}s'} c_{\vec{k}s} | \Psi \rangle$

even parity, spin singlet:

$$\hat{\Psi}_{\vec{k}} = \begin{pmatrix} 0 & f_0(\vec{k}) \\ -f_0(\vec{k}) & 0 \end{pmatrix} = i\hat{\sigma}^y f_0(\vec{k})$$

scalar wave function $f_0(\vec{k})$ with $f_0(-\vec{k}) = f_0(\vec{k})$ *even*

odd parity, spin triplet:

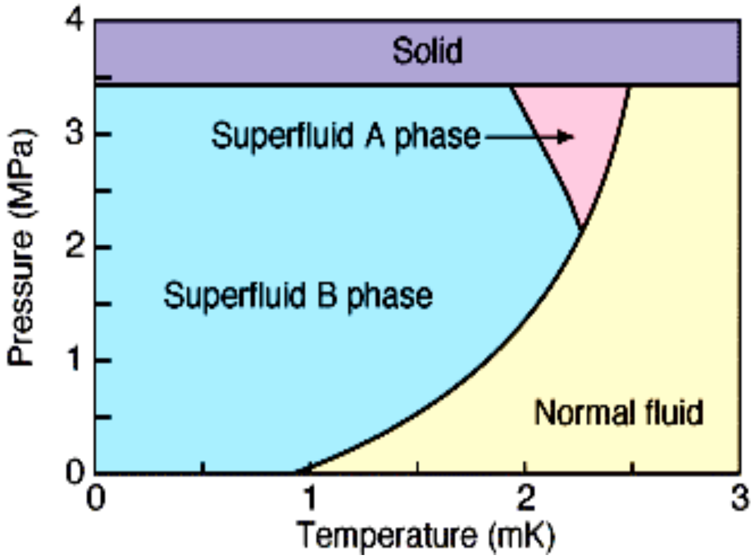
$$\hat{\Psi}_{\vec{k}} = \begin{pmatrix} -f_x + i f_y & f_z \\ f_z & f_x + i f_y \end{pmatrix} = i\vec{f}(\vec{k}) \cdot \hat{\vec{\sigma}} \hat{\sigma}^y$$

vector wave function $\vec{f}(\vec{k})$ with $\vec{f}(-\vec{k}) = -\vec{f}(\vec{k})$ *odd*

^{3}He - strongly correlated Fermi liquid

Superfluidity:

Lee, Osheroff & Richardson (1971)



Cooper pairing:
odd parity spin triplet

$\ell = 1 \qquad S = 1$

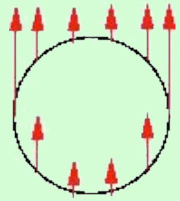
p-wave

Pair wavefunction:

$$\Psi_{\vec{k},ss'} = \begin{pmatrix} -f_x(\vec{k} + i f_y(\vec{k}) & f_z(\vec{k}) \\ f_z(\vec{k}) & f_x(\vec{k}) + i f_y(\vec{k}) \end{pmatrix}$$

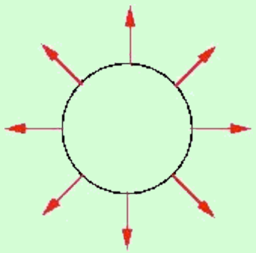
A-phase

$$\vec{f}(\vec{k}) = \hat{z}(k_x \pm i k_y)$$



B-phase

$$\vec{f}(\vec{k}) = \hat{x} k_x + \hat{y} k_y + \hat{z} k_z$$



Symmetry operations

Symmetries of normal phase: $\mathcal{G} = G_o \times G_s \times \mathcal{K} \times U(1)$

(G_o: orbital rotation; G_s: spin rotation; $\mathcal{K}$: time reversal; $U(1)$: gauge)

symmetry operation		
orbital rotation	$g c^\dagger_{\vec{k}s} = c^\dagger_{\hat{R}_o \vec{k},s}$	$\hat{R}_o$ orbital rotation
spin rotation	$g c^\dagger_{\vec{k}s} = \sum_{s'} D_{ss'} c^\dagger_{\vec{k}s'}$	$\hat{D} = e^{i\vec{\theta}\cdot\hat{\vec{\sigma}}/2}$
time reversal	$\hat{K} c^\dagger_{\vec{k}s} = \sum_{s'} (-i\hat{\sigma}^y)_{ss'} c_{-\vec{k}s'}$	
U(1) gauge	$\hat{\Phi} c^\dagger_{\vec{k}s} = e^{i\phi} c^\dagger_{\vec{k}s}$	vector potential $\vec{A}' = \vec{A} - \frac{\hbar c}{2e}\vec{\nabla}\phi$

presence of strong spin-orbit coupling → spin and lattice rotation go together

Symmetry operations

Symmetries of normal phase:

$$\underbrace{\mathcal{G}}_{} = \underset{\text{orbital rotation}}{G_o} \times \overset{\text{spin rotation}}{G_s} \times \underset{\text{time reversal}}{\mathcal{K}} \times \overset{\text{gauge}}{U(1)}$$

symmetry operation	spin singlet	spin triplet
orbital rotation	$g_o f_0(\vec{k}) = f_0(\hat{R}_o \vec{k})$	$g_o \vec{f}(\vec{k}) = \vec{f}(\hat{R}_o \vec{k})$
spin rotation	$g_s f_0(\vec{k}) = f_0(\vec{k})$	$g_s \vec{f}(\vec{k}) = \hat{R}_s \vec{f}(\vec{k})$
time reversal	$\hat{K} f_0(\vec{k}) = f_0(\vec{k})^*$	$\hat{K} \vec{f}(\vec{k}) = \vec{f}(\vec{k})^*$
U(1) gauge	$\hat{\Phi} f_0(\vec{k}) = f_0(\vec{k}) e^{i\phi}$	$\hat{\Phi} \vec{f}(\vec{k}) = \vec{f}(\vec{k}) e^{i\phi}$

presence of strong spin-orbit coupling → spin and lattice rotation go together

spin triplet: $g \vec{f}(\vec{k}) = \hat{R}_s \vec{f}(\hat{R}_o \vec{k})$ identical 3D rotations $\begin{cases} \hat{R}_o \\ \hat{R}_s \end{cases}$

Symmetry operations

Symmetries of normal phase: $\mathcal{G} = G_o \times G_s \times \mathcal{K} \times U(1)$

(spin rotation: G_s; gauge: $U(1)$; orbital rotation: G_o; time reversal: $\mathcal{K}$)

symmetry operation	spin singlet	spin triplet
orbital rotation	$g_o f_0(\vec{k}) = f_0(\hat{R}_o \vec{k})$	$g_o \vec{f}(\vec{k}) = \vec{f}(\hat{R}_o \vec{k})$
spin rotation	$g_s f_0(\vec{k}) = f_0(\vec{k})$	$g_s \vec{f}(\vec{k}) = \hat{R}_s \vec{f}(\vec{k})$
time reversal	$\hat{K} f_0(\vec{k}) = f_0(\vec{k})^*$	$\hat{K} \vec{f}(\vec{k}) = \vec{f}(\vec{k})^*$
U(1) gauge	$\hat{\Phi} f_0(\vec{k}) = f_0(\vec{k}) e^{i\phi}$	$\hat{\Phi} \vec{f}(\vec{k}) = \vec{f}(\vec{k}) e^{i\phi}$

Possible broken symmetries:

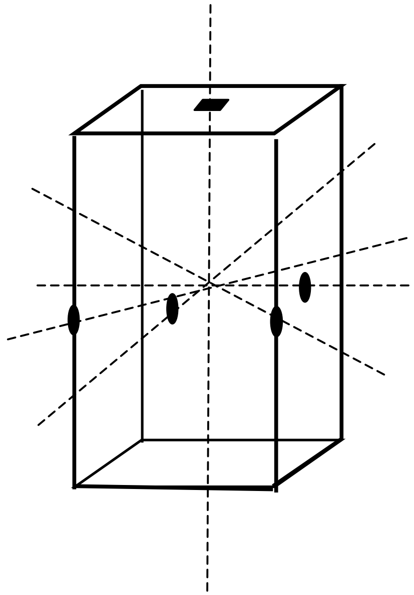
U(1)-gauge	*superconductivity*
orbital rotation	*crystal structure*
spin rotation	*magnetism*
time reversal	*magnetism*

Specific example:

Superconductor with
tetragonal crystal structure

Tetragonal crystal with spin orbit coupling

Point group: D_{4h}



D_{4h} contains inversion

→ even and odd representations

4 one-dim., 1 two-dim. representation

Character table for D_4

Γ	E	C_2	$2C_4$	$2C_2'$	$2C_2''$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	1	-1	1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

Tetragonal crystal with spin orbit coupling

Point group: D_{4h}

4 one-dim., 1 two-dim. representation
even (g) / odd (u) parity

Γ	$f_0(\vec{k})$	Γ	$\vec{f}(\vec{k})$
A_{1g}	1	A_{1u}	$\hat{x}k_x + \hat{y}k_y$
A_{2g}	$k_x k_y \left(k_x^2 - k_y^2\right)$	A_{2u}	$\hat{y}k_x - \hat{x}k_y$
B_{1g}	$k_x^2 - k_y^2$	B_{1u}	$\hat{x}k_x - \hat{y}k_y$
B_{2g}	$k_x k_y$	B_{2u}	$\hat{y}k_x + \hat{x}k_y$
E_g	$\{k_x k_z, k_y k_z\}$	E_u	$\{\hat{z}k_x, \hat{z}k_y\}$ $\{\hat{x}k_z, \hat{y}k_z\}$

Conventional: A_{1g}

Unconventional: everything else

only one representation is relevant for the superconducting phase transition

Ginzburg-Landau free energy functionals

1-dimensional representations: $f_0(\vec{k}) = \eta f_{0\gamma}(\vec{k})$, $\vec{f}(\vec{k}) = \eta \vec{f}_\gamma(\vec{k})$

$$F[\eta, \vec{A}] = \int d^3r \left[a|\eta|^2 + b|\eta|^4 + K|\vec{D}\eta|^2 + \frac{1}{8\pi}(\vec{\nabla} \times \vec{A})^2 \right]$$

like conventional SC

$$\vec{D} = \frac{\hbar}{i}\vec{\nabla} - \frac{2\pi}{\Phi_0}\vec{A}$$

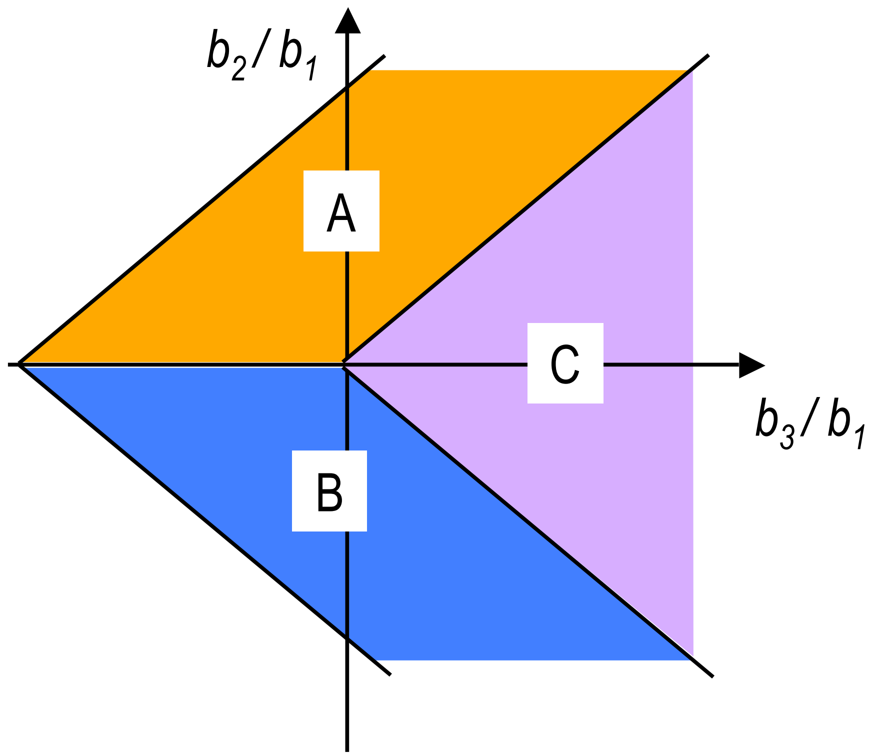
2-dimensional represenations: $f_0 = \eta_x f_{0x} + \eta_y f_{0y}$, $\vec{f} = \eta_x \vec{f}_x + \eta_y \vec{f}_y$

$$\begin{aligned} F[\vec{\eta}, \vec{A}] \quad &= \int d^3r \Big[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\left\{ \eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2} \right\} + b_3|\eta_x|^2|\eta_y|^2 \\ &+K_1\left\{ |D_x\eta_x|^2 + |D_y\eta_y|^2 \right\} + K_2\left\{ |D_x\eta_y|^2 + |D_y\eta_x|^2 \right\} \\ &+\left\{ K_3(D_x\eta_x)^*(D_y\eta_y) + K_4(D_x\eta_y)^*(D_y\eta_x) + c.c. \right\} \\ &+K_5\left\{ |D_z\eta_x|^2 + |D_z\eta_y|^2 \right\} + \tfrac{1}{8\pi}(\vec{\nabla} \times \vec{A})^2 \Big] \end{aligned}$$

Possible superconducting phases

Higher-dimensional order parameters are interesting $\vec{\eta} = (\eta_x, \eta_y)$

$$F[\vec{\eta}, \vec{A}] = \int d^3r \left[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\right\} + b_3|\eta_x|^2|\eta_y|^2 \right]$$



phase	$f_0(\vec{k})$	$\vec{f}(\vec{k})$	broken symmetry
A	$(k_x \pm i k_y)k_z$	$\hat{z}(k_x \pm i k_y)$	$U(1), \mathcal{K}$
B	$(k_x \pm k_y)k_z$	$\hat{z}(k_x \pm k_y)$	$U(1), D_{4h} \to D_{2h}$
C	$k_x k_z, k_y k_z$	$\hat{z}k_x, \hat{z}k_y$	$U(1), D_{4h} \to D_{2h}$

$\mathcal{K}$ → magnetism

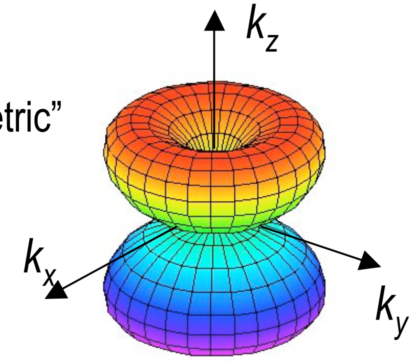
$D_{4h} \to D_{2h}$ → crystal deformation

Degeneracy: 2

domain formation possible

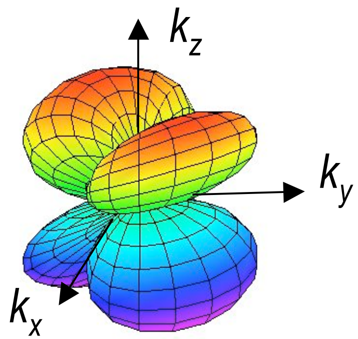
Phases

"axial symmetric"



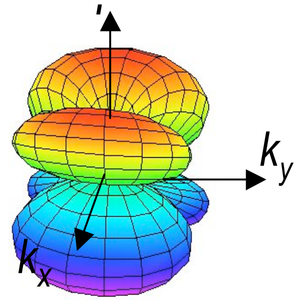
A-phase: $f_0(\vec{k}) = \eta k_z(k_x + ik_y)\ ,\ \ \eta k_z(k_x - ik_y)$

time reversal



B-phase: $f_0(\vec{k}) = \eta k_z(k_x + k_y)\ ,\ \ \eta k_z(k_x - k_y)$

C_4



C-phase: $f_0(\vec{k}) = \eta k_z k_x\ ,\ \ \eta k_z k_y$

C_4

"Weak-coupling argument":

less nodes → more stable ⇒ A-phase more stable than B/C-phase

Anisotropy

B- and C-phase violated crystal symmetry: *tetragonal* → *orthorhombic*

⇒ spontaneous crystal deformation *Tiny !*

Diamagnetic screening: supercurrents $\vec{j} = -c \frac{\partial F}{\partial \vec{A}}$

$$j_x = 8\pi e i \left[K_1 \eta_x^* D_x \eta_x + K_2 \eta_y^* D_x \eta_y + K_4 \eta_x^* D_y \eta_y + K_5 \eta_y^* D_y \eta_x - cc. \right]$$

$$j_y = 8\pi e i \left[K_1 \eta_y^* D_y \eta_y + K_2 \eta_x^* D_y \eta_x + K_4 \eta_y^* D_x \eta_x + K_5 \eta_x^* D_x \eta_y - cc. \right]$$

$$j_x = 8\pi e i K_3 \left\{ \eta_x^* D_z \eta_x + \eta_y^* D_z \eta_y - cc. \right\}$$

tensorial London equation: $\nabla^2 \vec{B} = \hat{\Lambda} \vec{B}$ Important for vortex lattice structure!

$$\hat{\Lambda}_A = \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix} \qquad \underbrace{\hat{\Lambda}_B = \begin{pmatrix} \lambda^{-2} & \tilde{\lambda}^{-2} & 0 \\ \tilde{\lambda}^{-2} & \lambda^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix} \quad \hat{\Lambda}_C = \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda'^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix}}$$

tetragonal orthorhombic